B.Sc. Part II 4th paper vector calculus

Q: Prove that

$$\text{div grad } r^n = n(n+1) r^{n-2}$$

and

$$\text{curl grad } r^n = 0.$$

Soln.

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ ——— (1)

$\Rightarrow r^2 = x^2 + y^2 + z^2$. ——— (2)

Differentiating (2) partially w.r.to x, we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{——— (3)}$$

Differentiating (2) partially w.r.to y, we get

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{——— (4)}$$

Differentiating (2) partially w.r.to z, we get

$$2r\frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r} \quad \text{——— (5)}$$

Let $\phi = r^n \Rightarrow \frac{\partial \phi}{\partial x} = n r^{n-1}\frac{\partial r}{\partial x} = n r^{n-1} \cdot \frac{x}{r}$ using (3)

$$\Rightarrow \frac{\partial \phi}{\partial x} = n x r^{n-2} \quad \text{——— (6)}$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(n x r^{n-2}\right) = n\left[\frac{\partial}{\partial x}\left(x r^{n-2}\right)\right]$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} = n\left[r^{n-2} + x\frac{\partial}{\partial x}\left(r^{n-2}\right)\right]$$

$$\therefore \frac{\partial^2\phi}{\partial x^2} = n\left[r^{n-2} + x\cdot(n-2)r^{n-3}\frac{\partial r}{\partial x}\right]$$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} = n\left[r^{n-2} + x(n-2)r^{n-3}\cdot\frac{x}{r}\right]$$

$$= n\left[r^{n-2} + \frac{x^2(n-2)\,r^{n-2}}{r^2}\right]$$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} = n\,r^{n-2}\left[1 + \frac{n-2}{r^2}x^2\right] \quad \text{--- (6)}$$

Similarly $\dfrac{\partial^2\phi}{\partial y^2} = n\,r^{n-2}\left[1 + \dfrac{n-2}{r^2}y^2\right]$ --- (7)

and $\dfrac{\partial^2\phi}{\partial z^2} = n\,r^{n-2}\left[1 + \dfrac{n-2}{r^2}z^2\right]$ --- (8)

Now, $\text{grad}\,\phi = \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$

$$\Rightarrow \text{grad}\,\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\Rightarrow \text{grad}\,\phi = \vec{i}\left(nx\,r^{n-2}\right) + \vec{j}\cdot ny\,r^{n-2} + \vec{k}\,nz\,r^{n-2}$$

$$= n\,r^{n-2}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) \quad [\text{using (6)}]$$

Now, $\text{div grad}\,\phi = \nabla\cdot(\nabla\phi) = \nabla^2\phi$

$$= \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)$$

$$\Rightarrow \text{div grad}\,\phi = n\,r^{n-2}\left[\left(1+\frac{n-2}{r^2}x^2\right) + \left(1+\frac{n-2}{r^2}y^2\right) + \left(1+\frac{n-2}{r^2}z^2\right)\right]$$

[using (6), (7) and (8)]

$$\Rightarrow \text{div grad } \phi = n r^{n-2}\left[3 + \frac{n-2}{r^2}(x^2+y^2+z^2)\right]$$

$$\Rightarrow \text{div grad } \phi = n r^{n-2}\left[3 + \frac{n-2}{r^2}\cdot r^2\right]$$

$$\Rightarrow \text{div grad } \phi = n \cdot r^{n-2}(3 + n-2)$$

$$\Rightarrow \text{div grad } \phi = n(n+1) r^{n-2} \quad \underline{\text{Proved}}$$

$$\Rightarrow \text{div grad } r^n = n(n+1) r^{n-2}$$

Now,

$$\text{curl grad } r^n = \text{curl grad } \phi$$

$$\Rightarrow \text{curl grad } r^n = \text{curl}\left(\vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \text{curl}\left(\vec{i}\cdot n x r^{n-2} + \vec{j} n \cdot y r^{n-2} + \vec{k}\, n z r^{n-2}\right)$$

$$= n\,\text{curl}\left(\vec{i}\, x r^{n-2} + \vec{j}\, y r^{n-2} + \vec{k}\, z r^{n-2}\right)$$

$$= n \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x r^{n-2} & y r^{n-2} & z r^{n-2} \end{vmatrix}$$

$$\Rightarrow \text{curl grad } r^n = n\Big[\vec{i}\left\{\frac{\partial}{\partial y}(z r^{n-2}) - \frac{\partial}{\partial z}(y r^{n-2})\right\}$$

$$- \vec{j}\left\{\frac{\partial}{\partial x}(z r^{n-2}) - \frac{\partial}{\partial z}(x r^{n-2})\right\}$$

$$+ \vec{k}\left\{\frac{\partial}{\partial x}(y r^{n-2}) - \frac{\partial}{\partial y}(x r^{n-2})\right\}\Big]$$

$$\Rightarrow \text{curl grad } \phi$$

$$= n\left[\vec{i}\left\{z(n-2)r^{n-3}\frac{\partial r}{\partial y} - y(n-2)r^{n-3}\frac{\partial r}{\partial z}\right\}\right.$$

$$-\vec{j}\left\{z(n-2)r^{n-3}\frac{\partial r}{\partial x} - x(n-2)r^{n-3}\frac{\partial r}{\partial z}\right\}$$

$$\left.+\vec{k}\left\{y(n-2)r^{n-3}\frac{\partial r}{\partial x} - x(n-2)r^{n-3}\frac{\partial r}{\partial y}\right\}\right]$$

$$\Rightarrow \text{curl grad } \phi = n\left[\vec{i}\left\{\cancel{z(n-2)r^{n-3}\cdot\frac{y}{r}} - \cancel{y(n-2)r^{n-3}\cdot\frac{z}{r}}\right\}\right.$$

$$-\vec{j}\left\{\cancel{z(n-2)r^{n-3}\cdot\frac{x}{r}} - \cancel{x(n-2)r^{n-3}\cdot\frac{z}{r}}\right\}$$

$$\left.+\vec{k}\left\{\cancel{y(n-2)r^{n-3}\frac{x}{r}} - \cancel{x(n-2)r^{n-3}\frac{y}{r}}\right\}\right]$$

$$\Rightarrow \text{curl grad } \phi = \underline{\underline{0}}.$$